Many-region vacuum entanglement: Distilling a W state

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We investigate the correlations between any number of arbitrarily far-apart regions of the vacuum of the free Klein-Gordon field by means of its finite duration coupling to an equal number of localized detectors. We show that the correlations between any N such regions enable us to distill an N-partite W state, and therefore exhibit true N-fold entanglement. Furthermore, we show that for $N = 3$, the correlations cannot be reproduced by a hybrid local-nonlocal hidden-variable model. For $N \geq 4$ the issue remains open.

In a recent paper [1], the nature of the correlations between two arbitrarily far-apart regions of the ground state of the free Klein-Gordon field was investigated by means of its finite duration coupling to a pair of localized detectors. It was shown that a local hidden-variable model cannot account for these correlations [1, 2], and that as a function of the separation between the regions, L, and the duration of the coupling, T, the entanglement decreases at a slower rate than $e^{-(L/cT)^3}$. It is, therefore, natural to ask whether the vacuum admits other types of these kinds of correlations, i.e. *true* many-region entanglement [3], and *full* nonlocality [4]. In this paper we will answer the first question affirmatively for any number of arbitrarily far-apart regions of the vacuum, N, while to the second question we will only be able to provide a positive answer for the case $N = 3$. We will follow the method developed in previous papers [1, 5], and begin by investigating in detail the three-region case. We will then use results thus obtained to study the correlations between any number of regions.

Our method consists of the finite duration coupling of the field we wish to investigate to any number of localized nonentangled detectors, such that all the detectors remain causally disconnected from one another throughout the interaction. Once the interaction is over, we trace over the field degrees of freedom to obtain the detectors' reduced density matrix. The crux of the method lies in the fact that since the detectors are initially nonentangled, any nonlocal correlations exhibited by the detectors' final reduced density matrix must have their origin in corresponding vacuum correlations. This enables us to apply recently developed tools from the field of quantum information theory to study the structure of the vacuum.

Before we begin, let us first give the definitions of true multi-fold entanglement and full nonlocality. A multipartite mixed state is said to be truly multi-fold entangled iff it cannot be expressed as a convex sum of decomposable terms. In the tri-partite case this just means that the state cannot be written as a convex sum of terms of the form $\rho_i \otimes \rho_{jk}$, where the subscripts denote any of the three subsystems. Examples of truly tri-fold entangled states are the GHZ [6] and W [7] states. Analogously, we can also distinguish between different types of nonlocality. A multi-partite state is fully nonlocal if there does not exist a decomposable hidden-variable dependent probability function that can account for the results of any set of von Neumann measurements. In the tri-partite case of a system composed of three parts A, B, and C, this means that $\wp_{ABC}(a,\, b,\, c \mid \lambda) \neq \wp_A(a \mid \lambda)\,\wp_{BC}(b,\, c \mid \lambda) + \wp_C(c \mid \lambda)\,\wp_{AB}(a,\, b \mid \lambda) + \wp_B(b \mid \lambda)\,\wp_{CA}(c,\, a \mid \lambda)$. Here λ is the hidden-variable, and $\wp_{ABC}(a,\, b,\, c \mid \lambda)$ is the probability for $\hat{a} = a$, $\hat{b} = b$, and $\hat{c} = c$. Otherwise, the state may admit a hybrid local-nonlocal hidden-variable description [8]. Svetlichny derived a Bell-like inequality to distinguish between these two cases [4], which was later generalized [9].

Let us consider the ground state of a free Klein-Gordon field and three nonentangled point-like two-level detectors [10]. The interaction Hamiltonian of the field and the detectors is given by

$$
\begin{aligned}
H_I(t) &= H_I^A(t) + H_I^B(t) + H_I^C(t) \\
&= \sum_{i=A,\,B,\,C} \int_{-T/2}^{t} dt' \epsilon_i(t') \left(e^{i\Omega_i t'} \sigma_i^+ + e^{-i\Omega_i t'} \sigma_i^- \right) \phi(\vec{x}_i,\, t'),
\end{aligned}
\tag{1}
$$

where $\phi(\vec{x},\, t)$ is a free Klein-Gordon field in three spatial dimensions, the $\sigma_i^\pm$ are the detectors' "ladder" operators,

and the Ω_i denote the energy gap between detector energy levels. T is the duration of the interaction, while the window-functions, $\epsilon_i(t)$, govern its strength. We set $cT << L_{ij}$, with $L_{ij} \equiv |\vec{x}_i - \vec{x}_j|$ and the $\vec{x}_i$ being the locations of the detectors. This ensures that the detectors remain causally disconnected throughout the interaction. The evolution operator therefore factors to a product. In the Dirac interaction representation, employing "natural" units ($\hbar = c = 1$), $U = \prod_{i=A,\,B,\,C} \hat{T} e^{-i\int dt H_I^i(t)}$, with $\hat{T}$ denoting time ordering. Expanding to the square order in the $\epsilon_i(t)$, once the interaction is over the final state of the system is given by

$$\begin{aligned} U|0\rangle|\downarrow\downarrow\downarrow\rangle \simeq\ & |0\rangle|\downarrow\downarrow\downarrow\rangle - i\Phi_A^+|0\rangle|\uparrow\downarrow\downarrow\rangle - i\Phi_B^+|0\rangle|\downarrow\uparrow\downarrow\rangle - i\Phi_C^+|0\rangle|\downarrow\downarrow\uparrow\rangle - \Phi_A^+\Phi_B^+|0\rangle|\uparrow\uparrow\downarrow\rangle - \Phi_B^+\Phi_C^+|0\rangle|\downarrow\uparrow\uparrow\rangle \\ & -\Phi_C^+\Phi_A^+|0\rangle|\uparrow\downarrow\uparrow\rangle - \sum_{i=A,\,B,\,C}\Theta_i|0\rangle|\downarrow\downarrow\downarrow\rangle + i\Phi_A^+\Phi_B^+\Phi_C^+|0\rangle|\uparrow\uparrow\uparrow\rangle + O\left(\epsilon^3\right), \end{aligned} \tag{2}$$

where $\Phi_i^{\pm} \equiv \int_{-T/2}^{T/2} dt \epsilon_i(t) e^{\pm i\Omega_i t}\phi(\vec{x}_i,\, t)$, and $\Theta_i \equiv \frac{1}{2}\hat{T}\left[\int_{-T/2}^{T/2} dt H_I^i(t) \int_{-T/2}^{T/2} dt' H_I^i(t')\right]$. (Actually the last term in the expansion is of cubic order, because unlike the other cubic terms, it cannot simply be discarded at this stage.) When working in the computational basis, $\{\downarrow\downarrow\downarrow,\, \downarrow\downarrow\uparrow,\, \downarrow\uparrow\downarrow,\, \downarrow\uparrow\uparrow,\, \uparrow\downarrow\downarrow,\, \uparrow\downarrow\uparrow,\, \uparrow\uparrow\downarrow,\, \uparrow\uparrow\uparrow\}$, the detectors' nonnormalized reduced density matrix is given by

$$\begin{pmatrix} 1-C & 0 & 0 & -d_{BC}^{++} & 0 & -d_{CA}^{++} & -d_{AB}^{++} & 0 \\ 0 & d_{CC}^{-+} & d_{BC}^{-+} & 0 & d_{CA}^{-+} & 0 & 0 & d_{ABCC}^{---+} \\ 0 & d_{BC}^{-+} & d_{BB}^{-+} & 0 & d_{AB}^{-+} & 0 & 0 & d_{ABCB}^{---+} \\ -d_{BC}^{++} & 0 & 0 & d_{BCBC}^{--++} & 0 & d_{CABC}^{--++} & d_{ABBC}^{--++} & 0 \\ 0 & d_{CA}^{-+} & d_{AB}^{-+} & 0 & d_{AA}^{-+} & 0 & 0 & d_{ABCA}^{---+} \\ -d_{CA}^{++} & 0 & 0 & d_{CABC}^{--++} & 0 & d_{CACA}^{--++} & d_{ABCA}^{--++} & 0 \\ -d_{AB}^{++} & 0 & 0 & d_{ABBC}^{--++} & 0 & d_{ABCA}^{--++} & d_{ABAB}^{--++} & 0 \\ 0 & d_{ABCC}^{---+} & d_{ABCB}^{---+} & 0 & d_{ABCA}^{---+} & 0 & 0 & d_{ABCABC}^{---+++} \end{pmatrix} + O\left(\epsilon^4\right). \tag{3}$$

Here we have employed the notation $d_{i\cdots n}^{\alpha\cdots\zeta} \equiv \langle 0|\,\Phi_i^{\alpha}\cdots\Phi_n^{\zeta}\,|0\rangle$, and $C \equiv \sum_i \langle\downarrow\downarrow\downarrow|\,\langle 0|\,\Theta_i\,|0\rangle\,|\downarrow\downarrow\downarrow\rangle$, with $i,\, \ldots,\, n = A,\, B,\, C$ and $\alpha,\, \ldots,\, \zeta = \pm$. For simplicity, we have chosen temporally symmetric window-functions. Hence the amplitudes are all real. d_{ii}^{-+} is the amplitude for a single photon emission by detector i, while $d_{i,\, j\neq i}^{++}$ is the amplitude for a single virtual photon exchange between detectors i and j. The physical interpretation of the rest of the amplitudes should thus be clear.

To prove that an N-partite mixed state is truly N-fold entangled, it is enough to show that it can be distilled to a truly N-fold entangled pure state [11, 12]. Having each of the detectors pass through a filter, which attenuates its "spin-down" component by a factor of η, the detectors' nonnormalized reduced density matrix is in the computational basis given by

$$\begin{pmatrix} \eta^6 & 0 & 0 & -\eta^4 d_{BC}^{++} & 0 & -\eta^4 d_{CA}^{++} & -\eta^4 d_{AB}^{++} & 0 \\ 0 & \eta^4 d_{CC}^{-+} & \eta^4 d_{BC}^{-+} & 0 & \eta^4 d_{CA}^{-+} & 0 & 0 & \eta^4 d_{ABCC}^{---+} \\ 0 & \eta^4 d_{BC}^{-+} & \eta^4 d_{BB}^{-+} & 0 & \eta^4 d_{AB}^{-+} & 0 & 0 & \eta^4 d_{ABCB}^{---+} \\ -\eta^4 d_{BC}^{++} & 0 & 0 & \eta^2 d_{BCBC}^{--++} & 0 & \eta^2 d_{CABC}^{--++} & \eta^2 d_{ABBC}^{--++} & 0 \\ 0 & \eta^4 d_{CA}^{-+} & \eta^4 d_{AB}^{-+} & 0 & \eta^4 d_{AA}^{-+} & 0 & 0 & \eta^4 d_{ABCA}^{---+} \\ -\eta^4 d_{CA}^{++} & 0 & 0 & \eta^2 d_{CABC}^{--++} & 0 & \eta^2 d_{CACA}^{--++} & \eta^2 d_{ABCA}^{--++} & 0 \\ -\eta^4 d_{AB}^{++} & 0 & 0 & \eta^2 d_{ABBC}^{--++} & 0 & \eta^2 d_{ABCA}^{--++} & \eta^2 d_{ABAB}^{--++} & 0 \\ 0 & \eta^4 d_{ABCC}^{---+} & \eta^4 d_{ABCB}^{---+} & 0 & \eta^4 d_{ABCA}^{---+} & 0 & 0 & d_{ABCABC}^{---+++} \end{pmatrix}. \tag{4}$$

Note that each of the components is written to its lowest nonvanishing order. The reason for this will shortly

become apparent. For $L_{ij} >> T$, the overlap amplitudes, $d^{-+}_{i,\,j\neq i}$, are negligible as compared to the emission, d^{-+}_{ii}, and exchange amplitudes, $d^{++}_{i,\,j\neq i}$. If we now take the window-function of detector C, and only detector C, to have a superoscillatory Fourier transform [13, 14], of a form as in [1, 15], then by a suitable choice of the remaining two window-functions and the L_{ij} the single-virtual photon exchange amplitudes involving detector C can be made arbitrarily larger than the rest [1], i.e., $d^{++}_{BC} = d^{++}_{CA} \gg \mathit{all\,other\,amplitudes}$. In this limit, if we set $\eta^2 = d^{++}_{BC} = d^{++}_{CA}$ and apply Wick's theorem [16], then it is readily seen that only amplitudes whose Wick decomposition contains terms consisting solely of single-virtual photon exchange amplitudes involving detector C survive the filtering. These amplitudes are of three types, representing single-virtual photon exchange processes $d^{\pm\pm}_{iC}$, double emission processes $d^{\mp\mp\pm\pm}_{iCiC}$, and the overlap between double emission processes by different detector pairs, $d^{\mp\mp\pm\pm}_{iCjC}$ $(j \neq i)$. The detectors' reduced density matrix is thus given by

$$\frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

This density matrix is pure and corresponds to the state $\frac{1}{\sqrt{3}}\left(|\downarrow\downarrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle\right)$, which, by means of local operations on each of the detectors, can be transformed into a W state, $\frac{1}{\sqrt{3}}\left(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\right)$. The W state is truly tri-fold entangled. Since the detectors are initially nonentangled and remain causally disconnected throughout the interaction, this entanglement must have its origin in corresponding vacuum entanglement, i.e., the vacuum is truly tri-fold entangled.

We now turn to the question of full nonlocality. The tri-partite W state violates the Svetlichny inequality [17], and is therefore fully nonlocal. Following the same reasoning as above, we conclude that the correlations between three arbitrarily separated regions of the vacuum do not admit a hybrid local-nonlocal hidden-variable description.

The generalization of the above analysis to any number of distinct regions of the vacuum, N, is straightforward. An N-partite W state [18] can always be distilled. This can be seen as follows. Allowing for trivial modifications resulting from the increase in the number of regions, we employ the same protocol as in the three-region case. We choose one and only one of the detectors, say detector N, to have the window-function with the superoscillating Fourier transform. Once the interaction is over, we have each of the detectors pass through a filter $\eta^2 = d^{++}_{AN} = d^{++}_{BN} = \ldots = d^{++}_{N-1,\,N}$. (For any set of the L_{ij}, such a filter can be realized, as it is only the L_{iN} that have to be taken into consideration, while the the $\epsilon_i\left(t\right)$ can always be suitably adjusted.) Only amplitudes whose Wick decomposition contains terms consisting solely of single-virtual photon exchange amplitudes, involving detector N, survive the filtering. These amplitudes are of the same three types mentioned earlier. It is now only a matter of diligent book-keeping of the indices to convince oneself that, not excluding local operations, the filtering leaves us with an N-partite W state. It follows that the correlations between any N arbitrarily separated regions of the vacuum exhibit true N-fold entanglement.

As regards nonlocality, the situation is different. For $N \geq 4$, numerical computations indicate that N-partite Svetlichny-type inequalities [11] are not violated by the N-partite W state [19]. This, of course, does not mean that the correlations between $N \geq 4$ distinct regions are not fully nonlocal. It may be that stronger inequalities, e.g. generalizations of the N-partite Svetlichny-type inequalities incorporating more than two measurement settings per system [20], will reveal the N-partite W state as fully nonlocal. Or a different detection scheme may make possible the distillation of states which are known to be fully nonlocal, e.g. multi-partite GHZ states. At present this question remains unresolved.

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